JOURNAL OF AIRCRAFT
Vol. 38, No. 4, July–August 2001

Accurate Subcritical Damping Solution of Flutter Equation Using Piecewise Aerodynamic Function

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An alternative to the *p–k* method for accurate subcritical damping solution is presented. By combining the concepts used in rational function aerodynamics and using the piecewise aerodynamic interpolation function commonly used for the *k* method, a piecewise aerodynamic flutter equation is defined, which provides for accurate noncritical damping. A solution of the piecewise flutter equation is described, which is noniterative and which can be used to get results at a single velocity. The resulting flutter method provides a solution of the flutter equation with accurate subcritical damping that is efficient and reliable.

Nomenclature

A_m, β_{m-2}	=	coefficients in the rational function approximation (RFA)
A_m^j	=	piecewise quadratic generalized aerodynamic force interpolation function coefficients
b_0	=	reference length
g	=	added structural damping for k method
$Im()$	=	imaginary part of complex number
i	=	imaginary constant = $sqrt(-1)$
k	=	reduced frequency $(b_0\omega/V)$
k_j, k_{j+1}	=	range of reduced frequency
M, K	=	generalized mass and stiffness
$Q(k)$	=	generalized aerodynamic forces as computed by the doublet lattice method
$\hat{Q}_{ik}(ik)$	=	RFA for generalized aerodynamic forces
$\hat{Q}_k^j(k)$	=	piecewise quadratic generalized aerodynamic forces interpolation function
$\hat{Q}_s(s)$	=	RFA for generalized aerodynamic forces in the Laplace domain
$\hat{Q}_s^j(s)$	=	piecewise quadratic generalized aerodynamic forces function
q	=	generalized coordinate
s, p	=	Laplace variable
V, ρ	=	velocity and density
ω	=	circular frequency

Introduction

FLUTTER is instability caused by the coupling of aerodynamic forces with the structural forces of a flexible structure. There are three principal interests in analysis of the flutter phenomenon. The first is the calculation of the velocity at the onset of flutter. The second is the trend of the modal frequency and damping characteristics vs velocity before the onset of flutter. The third interest in flutter is the interaction of these forces with a control system.

The velocity at the onset of flutter is the point at which the coupling of the structural forces and the aerodynamic forces cause the system to become unstable. The k method flutter solution[1] is the most common technique used for this analysis. This method is very reliable and cost effective. A spline is typically used to interpolate the aerodynamic forces for a large list of reduced frequencies. A solution is computed at each reduced frequency, and the roots must then be sorted. However, for this method, only the flutter crossing is correct. The frequency and damping values before and after the crossing are not reliable.

Subcritical frequency and damping results can be compared with measured data to evaluate the accuracy of the analytical models and thus the accuracy of the predicted flutter velocity. Also, an experienced flutter analyst may draw conclusions from the frequency and damping vs velocity curves that lead to structural modifications to extend the flutter velocity. In both cases, the frequency and damping values should be as accurate as possible.

Two methods are most commonly used to obtain accurate subcritical damping for a flexible aircraft, the p–k method, and transient flutter. The p–k method[2] repeatedly interpolates for the required aerodynamics as it iterates to find each eigenvalue. A spline or other function is defined to interpolate the aerodynamics to the required reduced frequency, which is determined from the eigenvalue from the previous iteration. The p–k method is more costly than the k method and problems with convergence can occur. Transient flutter uses a rational function approximation (RFA) of the aerodynamic forces.[3] This method tries to define a single function to represent the aerodynamic forces over a wide range of reduced frequency. This step can be difficult and time consuming. However, once the RFA is defined the flutter solution is straightforward.

The interaction of an aeroelastic system with a control system is known as aeroservoelasticity (ASE). The most common approach to ASE is to create a linear differential equation, in state-space form, of the aeroelastic dynamics so that it can be easily coupled with a linear model of the control system, also in state-space form. There are three primary methods used to create this model. The first and most common method is based on RFA aerodynamics, as in transient flutter. For ASE the definition of the system inputs (control surfaces) and outputs (control sensors) is additionally required. The second method is implemented in the FAMUSS (Flexible Aircraft Modeling Using State Space) program,[4] and uses an equivalent system method to create the ASE model. This technique uses an eigenvalue solution of the flutter equation with accurate frequency and damping to define the dynamic matrix of the state-space model. The rest of the ASE model is defined by a fit of the transfer function responses. The third method is the P-transform method.[5] It also requires a flutter solution with accurate frequency and damping to define the state-space dynamic matrix and makes use of the flutter eigenvectors to create the rest of the ASE model.

Background on Various Flutter Solution Methods

The basic aeroelastic equations of motion are given by Eq. (1). It is usually defined in generalized or modal coordinates to reduce the number of degrees of freedom.

$$M\ddot{q} + Kq - \tfrac{1}{2}\rho V^2 Q(k)q = 0 \tag{1}$$

The aerodynamic forces $Q(k)$ are a function of reduced frequency. The use of these aerodynamics is in and of itself an

Received 14 September 1999; revision received 15 October 2000; accepted for publication 7 November 2000.

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approximation because they are only valid for oscillatory motion. Also, they are typically computed at a list of discrete reduced frequencies. Thus, a direct solution of Eq. (1) is not possible. To solve it, some approximation of these aerodynamics must be done. The three commonly used solutions are the k method, the p–k method, and transient flutter. The primary differences in these methods are the type of damping and the treatment of the aerodynamics.

k Method Flutter

The k-method flutter equation is given by Eq. (2). The damping here is added structural damping, the inertia and aerodynamic forces are for purely oscillatory motion.

$$\left\{M - [(1+ig)/\omega^2]K + \left(\rho b_0^2 / 2k^2\right)Q(k)\right\}q = 0 \qquad (2)$$

The aerodynamic forces $Q(k)$ are typically computed for a small set of discrete reduced frequencies and then interpolated to a larger set of reduced frequencies. The equation is then solved for the eigenvalue $\lambda = (1+ig)/\omega^2$ at each interpolated reduced frequency. The frequency and damping characteristics come directly from the eigenvalue, but the velocity is computed from the reduced frequency given the eigenvalue frequency.

p–k Method Flutter

The p–k method flutter equation is given by Eq. (3). The damping here is a rate-of-decay type but is only applied to the inertia forces. The inertia forces are for damped motion, but the aerodynamic forces are treated as complex stiffness and therefore are only for purely oscillatory motion.

$$\left[Mp^2 + K - \tfrac{1}{2}\rho V^2 Q(k)\right]q = 0 \qquad (3)$$

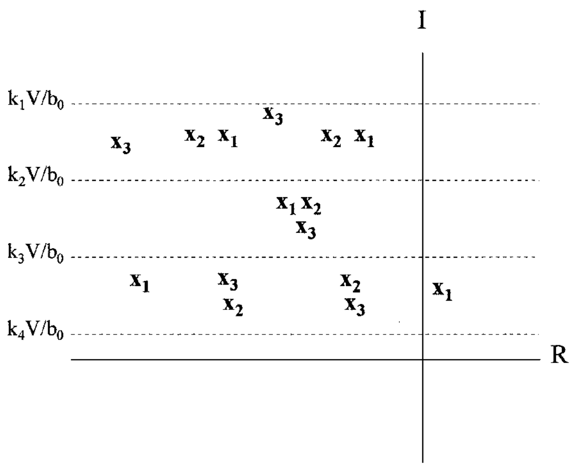

Fig. 1a Roots x_j from all of the piecewise flutter equations (where the aerodynamic forces are valid from k_j to k_{j+1}) at a single velocity.

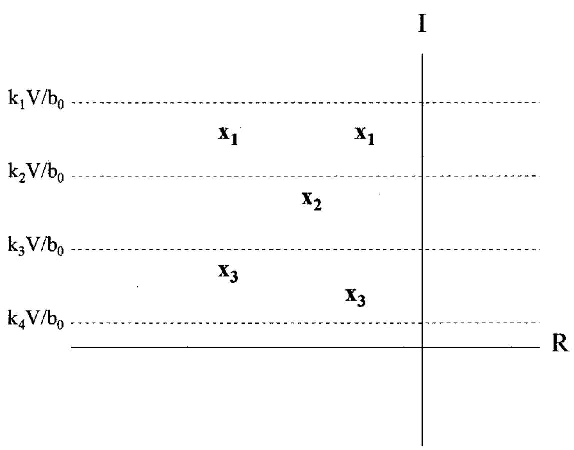

Fig. 1b Collection of roots with valid frequencies at a single velocity.

An iterative scheme is used to find each eigenvalue at a given velocity and density. At each iteration new aerodynamic forces are obtained using a spline or other interpolation function based on the eigenvalue frequency from the earlier iteration. The iterations continue until the eigenvalue frequency converges. This iterative method requires an initial estimate of the imaginary part of each eigenvalue. These are typically determined by extrapolating from results at preceding velocities. The natural frequencies are used to estimate the reduced frequency at the first velocity.

Transient Flutter Method

The transient flutter equation [Eq. (4)] is effectively the direct Laplace transformation of Eq. (1). Thus the damping is true Laplace damping, and both the inertia forces and the aerodynamic forces are for damped motion.

$$\left[Ms^2 + K - \tfrac{1}{2}\rho V^2 \hat{Q}_s(s)\right]q = 0 \qquad (4)$$

This equation requires the unsteady aerodynamic forces to be expressed in the Laplace domain, which are not typically available and are therefore approximated. An RFA is used for the aerodynamic forces. The RFA defines an equation that describes the aerodynamic forces over a wide range of frequency. A typical RFA uses a quadratic with additional lag terms. A fit of a set of known aerodynamics determines the coefficients of the equation. Once determined, the RFA equation, which was based on oscillatory aerodynamic forces, is used as an approximation of the aerodynamic forces for damped motion. At this point the solution of the flutter equation is straightforward.

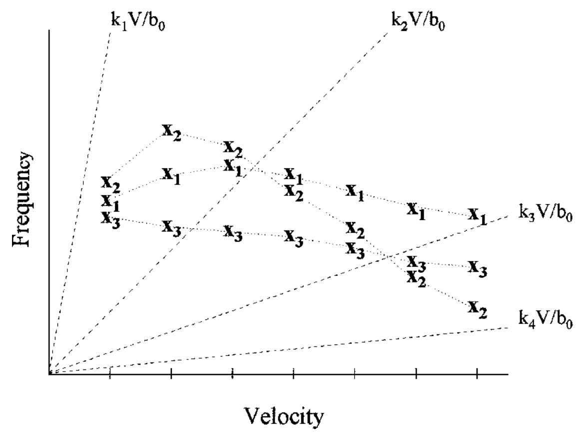

Fig. 2a Tracking a single mode (root x_j) as velocity varies from all of the piecewise flutter equations (where the aerodynamic forces are valid from k_j to k_{j+1}).

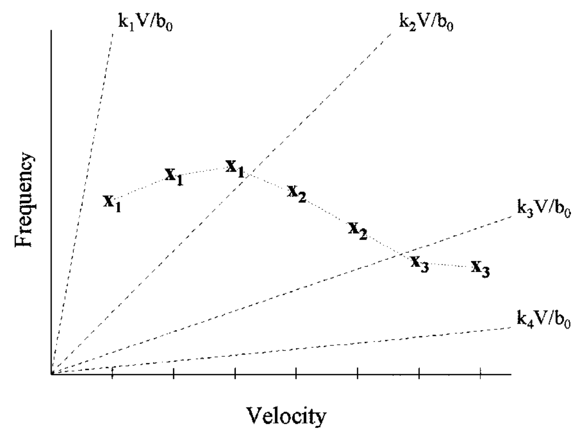

Fig. 2b Collection of roots with valid frequencies that have been tracked as velocity varies.

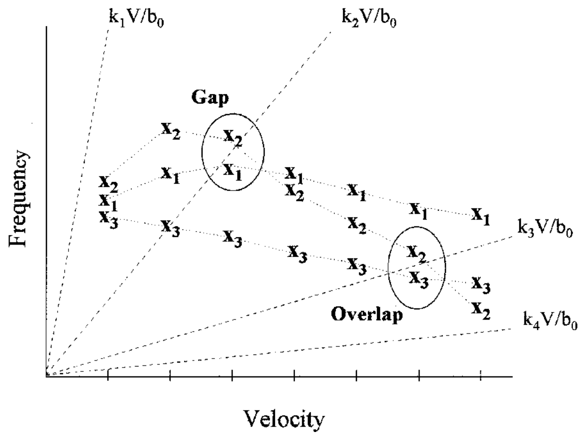

Fig. 3 Discontinuity in the flutter equation from one range of reduced frequency to the next may cause overlaps and gaps.

Background on the Various Treatments of the Aerodynamics

Each of the flutter solution methods just discussed treats the aerodynamics in a unique way. Each method must do this because the unsteady aerodynamics are not typically available in the Laplace domain. The most common method of producing the aerodynamic forces is the doublet lattice method,[6] which is only valid for oscillatory motion. This method defines the aerodynamic forces at specific reduced frequencies. Because of the significant computer resource costs, the list of reduced frequencies at which they are computed is typically rather small, around 10. Cunningham and Desmarais generalized the subsonic unsteady aerodynamic kernel function into the Laplace domain.[7] Their method provides a means of defining aerodynamic forces for damped motion, but they must be computed over a two-dimensional space, frequency and damping, which greatly increases the computational costs.

Piecewise Interpolation Function

One treatment of the aerodynamics defines an interpolation function that can be used to approximate the aerodynamics at any reduced

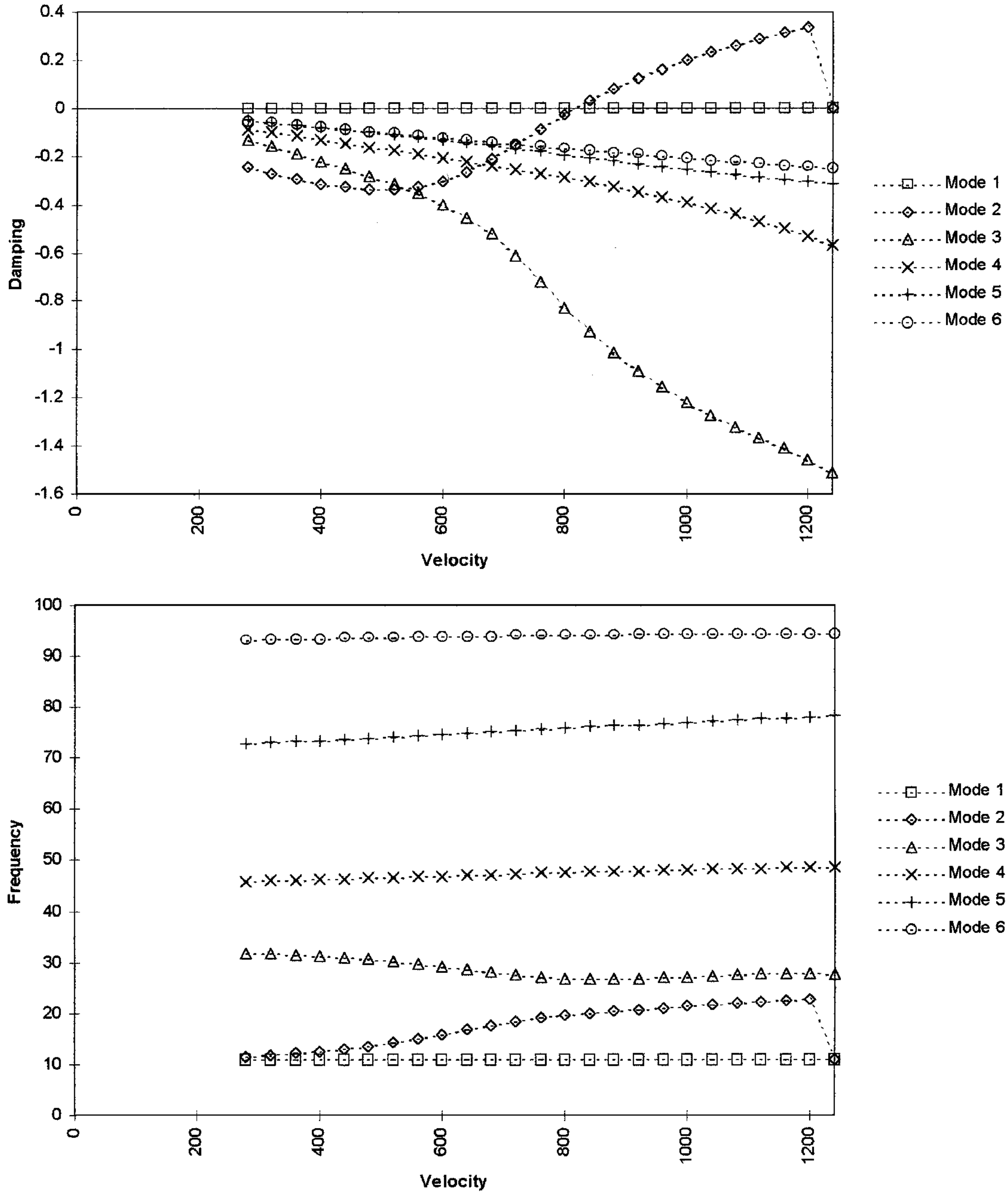

Fig. 4 *p–k* flutter solution for the cantilevered lifting surface test case at 0.9 Mach with six symmetric modes.

frequency within the original range. Generally the interpolation function is a piecewise quadratic or cubic spline defining an interpolation function for each interval of reduced frequency. This is commonly used in both k-method flutter solutions and p–k method solutions.

Equation (5) gives a typical quadratic piecewise interpolation function for a k-method flutter solution. The coefficients for each interval of reduced frequency are computed using four points, two on each side, and is referred to as an averaged quadratic.

$$\hat{Q}^j_k(k) = \sum_{m=0}^{2}(k)^m A^j_m, \qquad k_j \le k \le k_{j+1} \tag{5}$$

Each function matches the aerodynamic forces at the end points of its interval. The first and last intervals use three points. Unlike a cubic spline no constraints are placed on the function derivatives.

Rational Function Approximation

Another treatment of the unsteady aerodynamic forces is with an RFA. This technique is similar to the interpolation function just described except this method defines a single function that approximates the unsteady aerodynamic forces over the entire range of reduced frequencies for which they have been computed. Lag terms are added to enable the function to be used for the entire reduced frequency range. The coefficients are typically determined by a least-squares fit of the computed aerodynamic forces. It may be difficult to define an equation that fits the aerodynamic forces adequately over the entire range of reduced frequency.

Equation (6) gives a typical RFA representation of the unsteady aerodynamics as a function of reduced frequency (with imaginary value included). Key to the accuracy of the fit is the number of lag terms and the choice of lag constants β_{m-2}.

$$\hat{Q}_{ik}(ik) = \sum_{m=0}^{2}(ik)^m A_m + \sum_{m=3}^{6}\frac{(ik)A_m}{(ik)+\beta_{m-2}} \tag{6}$$

Other RFA representations optimize the lag constants or use a state-space notation that defines a lag matrix. The RFA function is still only valid for purely oscillatory motion. Regardless of the form of the RFA, it can be used to approximate the unsteady aerodynamics in the Laplace domain by substituting the scaled complex Laplace variable (sb_0/V) for the imaginary value (ik) [Eq. (7)].

$$\hat{Q}_s(s) = \sum_{m=0}^{2}s^m\left(\frac{b_0}{V}\right)^m A_m + \sum_{m=3}^{6}\frac{(s)A_m}{(s)+\beta_{m-2}(V/b_0)} \tag{7}$$

The substitution of a scaled Laplace variable for the reduced frequency is used for both ASE and transient flutter. This substitution

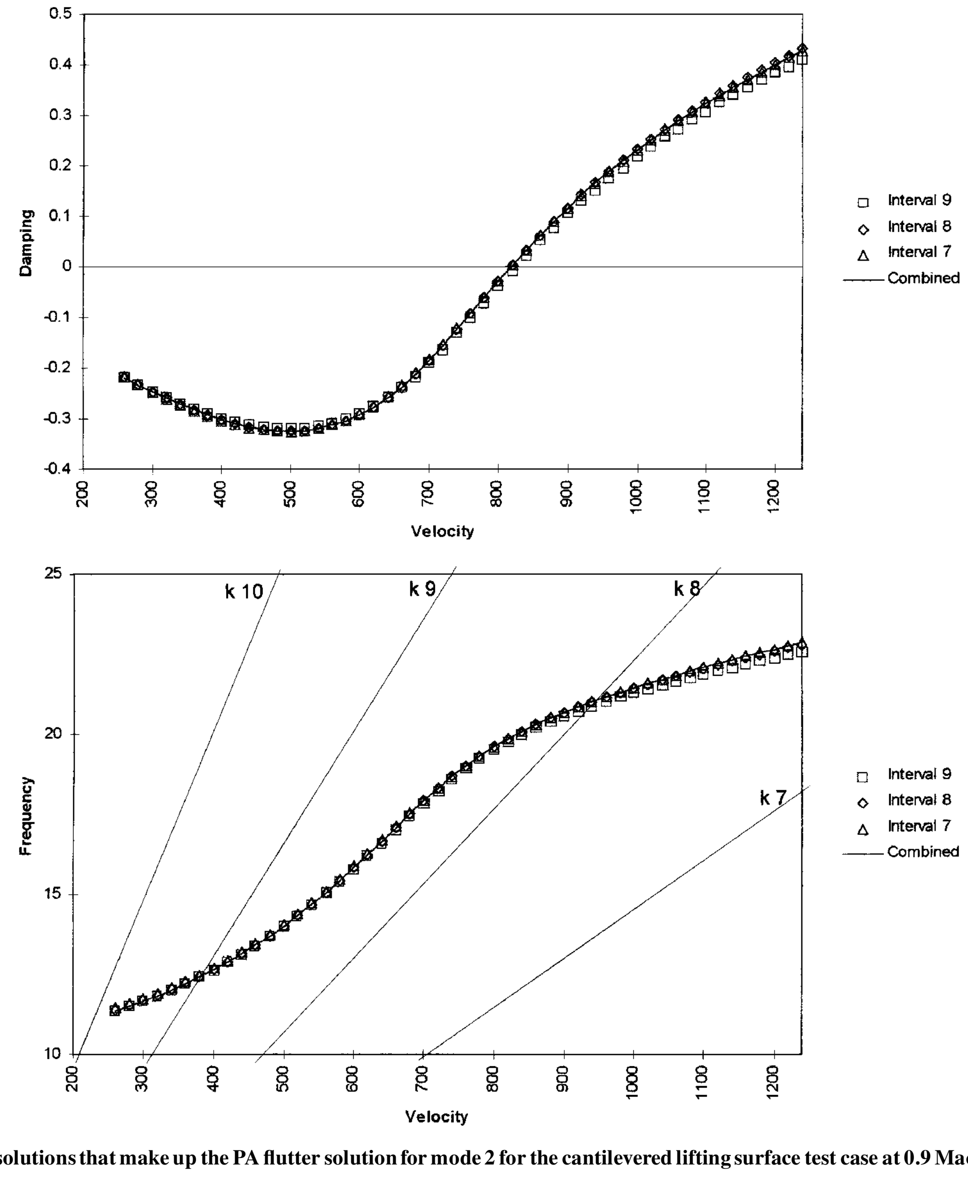

Fig. 5a Piecewise solutions that make up the PA flutter solution for mode 2 for the cantilevered lifting surface test case at 0.9 Mach with six symmetric modes.

was determined by Cunningham and Desmarais to be fairly good for low reduced frequencies and low damping.[7] It approximates the aerodynamic forces for damped motion from those computed for oscillatory motion. This differs somewhat from simply using the oscillatory aerodynamic forces for damped motion as done in the p–k flutter method.

Piecewise Aerodynamic Flutter Method

The piecewise aerodynamic (PA) flutter method makes use of concepts utilized in the k-method flutter solution, in transient flutter using RFA and in the p–k flutter method. The PA interpolation function used for the k method defines the aerodynamic forces for each interval of reduced frequency. Each equation, one for each range of reduced frequency, is used to approximate the aerodynamic forces in the Laplace domain as done with RFA aerodynamics for transient flutter. The roots of each of these equations are solved with generic matrix methods. The roots with a frequency within the range for which the aerodynamics were valid are accepted as done in the p–k method.

A substitution similar to that used for RFA [Eqs. (6) and (7)] is applied to each k-method aerodynamic interpolation function [Eq. (5)] to define the aerodynamic forces in the Laplace domain. Equation (8) defines these approximated aerodynamic forces in the Laplace domain by substituting the scaled complex Laplace variable sb_0/iV for the real reduced frequency k. Each of these functions is effectively an RFA with no lags. No lag terms are required because the function must only match the computed aerodynamic forces at two reduced frequencies, but each is only valid for the specified range of frequency. The imaginary portion of the Laplace variable, frequency of oscillation, is used to determine when the aerodynamic forces are valid.

$$\hat{Q}_s^j(s) = \sum_{m=0}^{2} s^m \left(\frac{b_0}{iV}\right)^m A_m^j, \qquad k_j \leq \frac{Im(s)b_0}{V} \leq k_{j+1} \tag{8}$$

This aerodynamic force function is inserted into the Laplace flutter equation used for transient flutter [Eq. (4)] resulting in Eq. (9). Now we have a series of Laplace domain equations each of which are valid for frequency of oscillations within a range defined by the interval of reduced frequency.

$$\Big\{\big[M - \tfrac{1}{2}V^2(b_0/iV)^2 A_2^j\big]s^2 - \tfrac{1}{2}V^2(b_0/iV)A_1^j(s) + K - \tfrac{1}{2}V^2 A_0^j\Big\}q = 0, \qquad k_j \leq Im(s)b_0/V \leq k_{j+1} \tag{9}$$

When each of these equations is solved for its roots, only those roots with frequencies within the range of reduced frequency for

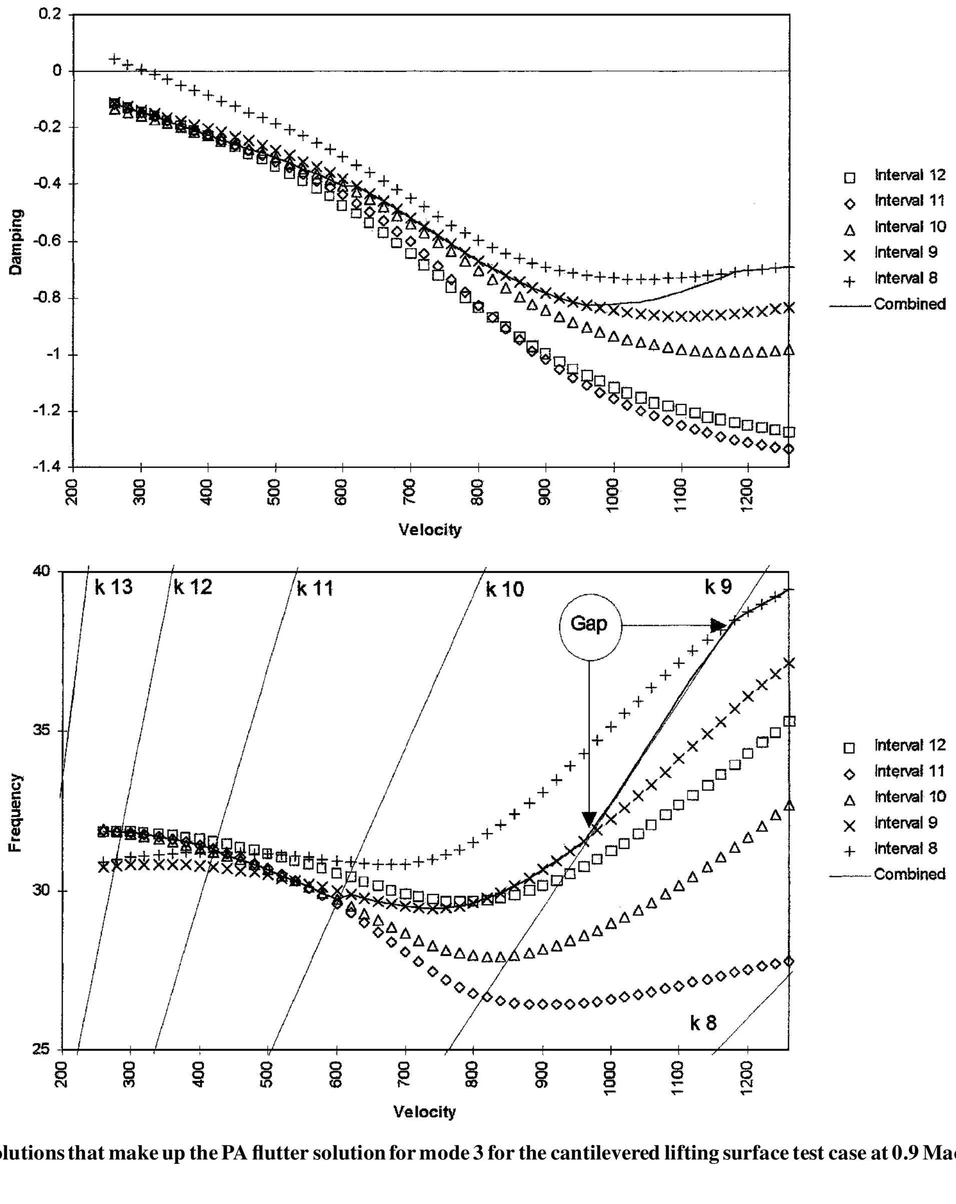

Fig. 5b Piecewise solutions that make up the PA flutter solution for mode 3 for the cantilevered lifting surface test case at 0.9 Mach with six symmetric modes.

which the aerodynamics are valid are in fact a solution to the equation. This is similar to the *p–k* method that iterates until a root is found that has a frequency matching the reduced frequency of the aerodynamics, which were used. The difference in the PA method is that all of the roots for each piecewise flutter equation are computed using generic matrix methods, which do not require iteration, and then only those roots in the proper frequency range are accepted. Thus all roots are found with N-1 eigenvalue solutions, where N is the number of reduced frequencies for which the aerodynamic forces were computed.

The rest of this section uses fictionalized data created for the purpose of demonstrating the PA flutter method and do not represent an actual test case.

Figure 1a shows a collection of roots plotted in the complex plane from the solution of each of three piecewise flutter equations. In this fictional example there are five modes. Each equation is only valid for a specific range of reduced frequency. This range of reduced frequency scales to a range of frequency (imaginary part of each root) for a particular velocity. Figure 1b shows only the roots from each solution, which have a frequency within the range for which the aerodynamic force function was valid.

This technique allows the roots at a single velocity to be obtained without data from earlier velocities. Another way to look at the same data is illustrated in Figs. 2a and 2b. In this case the root frequencies for a single "mode" are plotted vs velocity. There is a corresponding damping curve not shown. A mode implies a tracking operation to determine how to connect a root from one velocity to the next. One tracking scheme extrapolates to determine the expected eigenvalue at the next velocity and then matches the actual eigenvalue that is closest to the expected value. This technique is used when a complete flutter solution is desired, not just the roots at a single velocity. Note that for a given mode the curve computed from each piecewise flutter equation will begin at nearly the same point (they all start at the natural frequency at zero velocity). Figure 2a shows the roots for a given mode for all velocities and all of the piecewise flutter equations. Figure 2b shows only the roots, which have a frequency within the range for which the aerodynamic force function was valid.

The piecewise aerodynamic force functions are continuous from one range of reduced frequency to the next, but the quadratic function used to define them changes. Thus the eigenvalues of each of the piecewise flutter equations are not necessarily continuous, i.e., the eigenvalues for each function at the crossover reduced frequency may differ. This can cause two problems: overlaps and gaps. Figure 3 illustrates these two problems graphically. Both of these situations are easily handled by accepting the eigenvalues from both and averaging the eigenvalues. A simple method to determine the weighting factors is based on the number of points involved. With one point the factors are $(\frac{1}{2}, \frac{1}{2})$, with two points, the factors are $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{2}{3})$, etc.

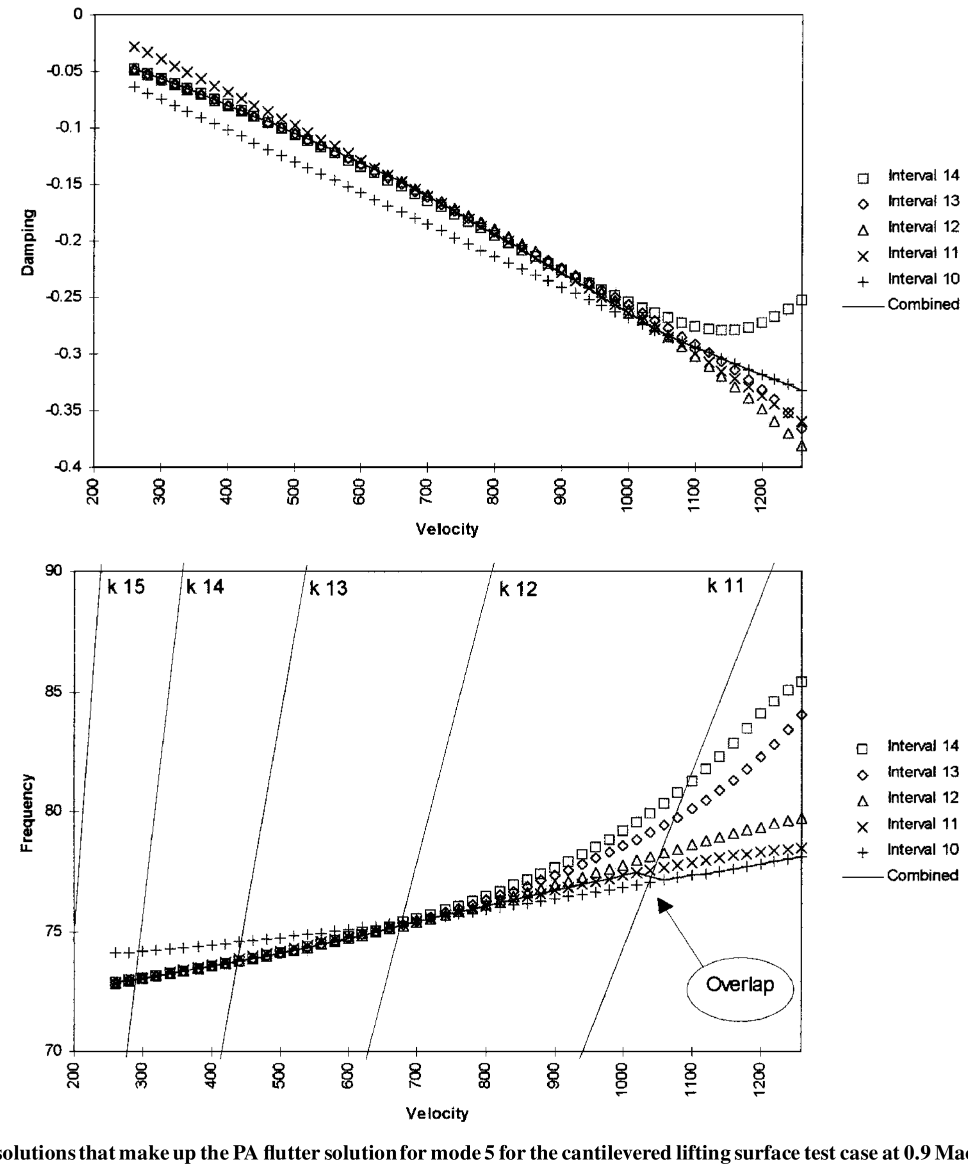

Fig. 5c Piecewise solutions that make up the PA flutter solution for mode 5 for the cantilevered lifting surface test case at 0.9 Mach with six symmetric modes.

Results

The PA flutter method was applied to a cantilevered lifting surface with symmetric boundary conditions. The analysis was done for 0.9 Mach at sea level with six flexible modes. The lifting surface planform has a span of 96.1 in. (244.1 cm), a root chord of 112.3 in. (285.2 cm), and a tip chord of 40.8 in. (103.6 cm). It has a sweep angle of 44.1 deg. The lifting surface was modeled with 13 boxes in the spanwise direction and 18 boxes in the chordwise direction. A reference chord of 85.28 in. (216.6 cm) was used in the unsteady aerodynamic computations. The aerodynamic influence coefficients were computed for 11 reduced frequencies ranging from 0.8 to 5.0. The six flexible modes have natural frequencies of 10.75, 11.10, 31.78, 45.75, 72.65, and 93.21 Hz. A p–k flutter analysis was also done on this model for comparison. The p–k solution is shown in Fig. 4. There is a tracking error in the p–k solution for mode 2. Iterated to mode 1, and thus two solutions were obtained for mode 1 at that velocity.

Each mode in a PA flutter solution is made up from the solution of several piecewise flutter equations, each of which is only valid for a specific range of reduced frequency. The solutions that make up modes 2, 3, and 5 are shown in Figs. 5a–5c. The lines for the reduced frequency ranges are also shown. A "gap" occurs for mode 3 in Fig. 5b, and an "overlap" occurs in Figure 5c. Again these may occur in the transition from one piecewise flutter equation to the next. The averaging scheme does a good job in the transition. These gaps and overlaps are caused by discontinuities in the piecewise aerodynamic functions. The discontinuities are affected by the choice of the reduced frequencies for which the aerodynamics are computed. This choice is important for all flutter methods and is especially important for the PA flutter method.

A direct comparison between the PA flutter solution and the p–k flutter solution for modes 2, 3, and 5 is given in Fig. 6. The difference between the two solutions increases where there is more damping. This is because of the difference in the representation of the aerodynamic forces. The aerodynamic forces in the p–k flutter equation are the same for damped motion as for oscillatory motion. The aerodynamic forces in the PA flutter equation approximate the forces for damped motion as is done for transient flutter with RFA aerodynamics.

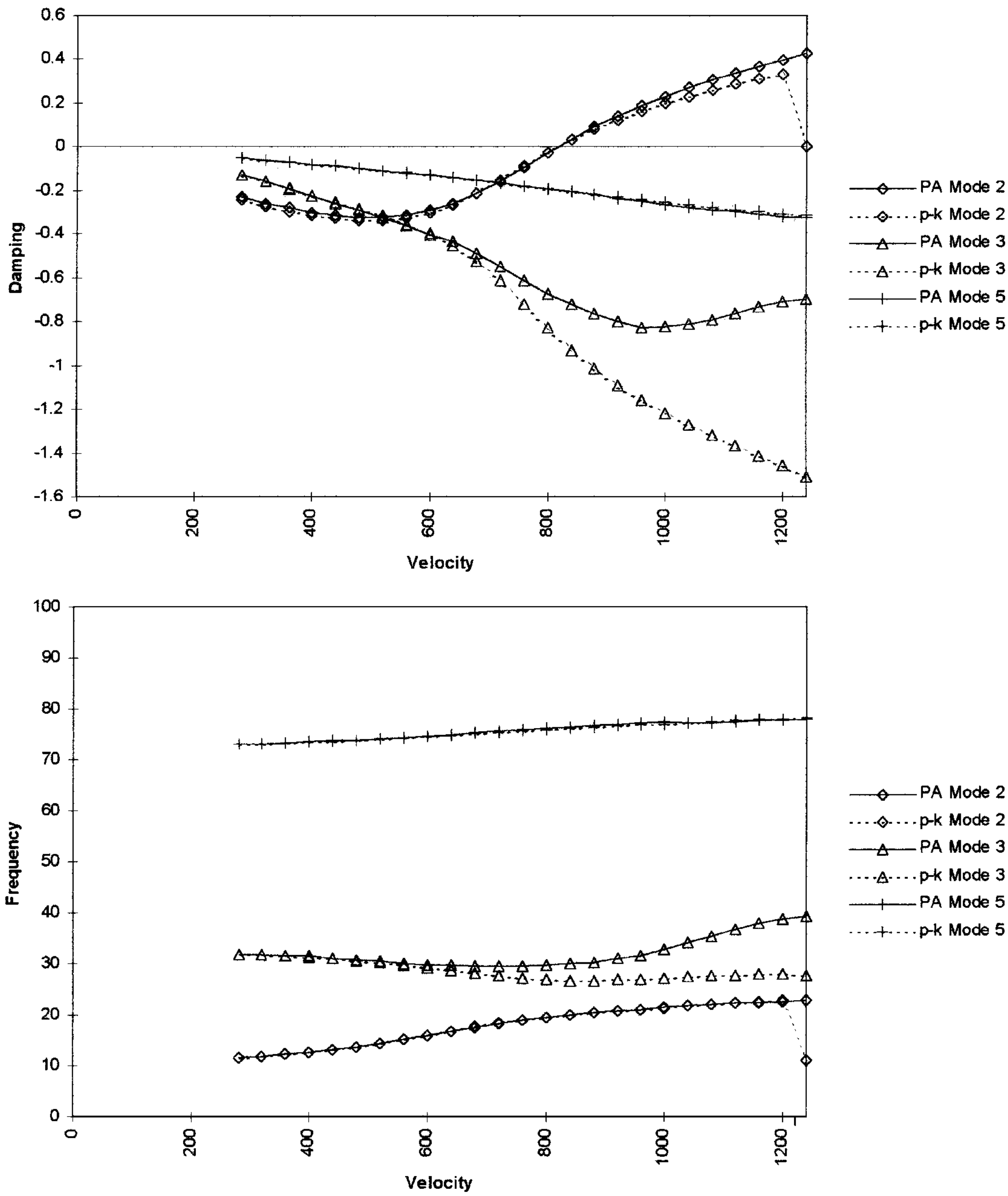

Fig. 6 Comparison of the p–k flutter solution and the PA flutter solution for modes 2, 3, and 5 for the cantilevered lifting surface test case at 0.9 Mach with six symmetric modes.

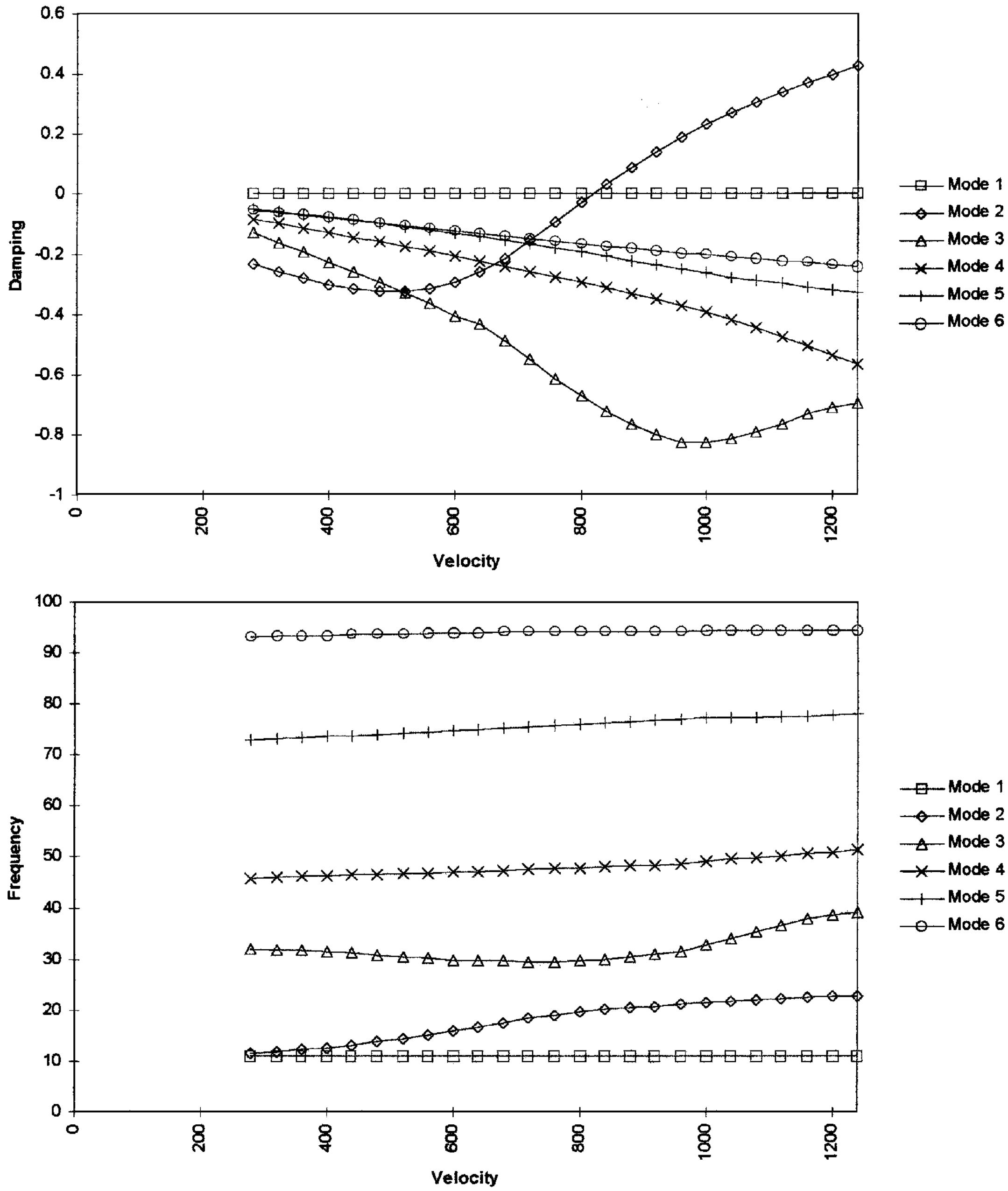

Fig. 7 PA flutter solution for the cantilevered lifting surface test case at 0.9 Mach with six symmetric modes.

The complete PA flutter solution is given in Fig. 7. The character of the PA flutter solutions (the shape of the frequency and damping curves) is very similar to the p–k solution as is to be expected. The flutter crossing ($g = 0.0$) for the two methods differ by only 0.025% in velocity and 0.1% in frequency. The CPU time required for the PA solution was approximately 25% of the CPU time required for the p–k solution. These savings in computational time are achieved by virtue of the noniterative nature of the PA flutter method. All of the required data are computed with a fixed number of complex eigenvalue solutions. The individual solutions are sorted so that each mode is "tracked" as velocity varies. Then it is determined which parts from each solution are valid for a given mode. At the same time the overlaps and gaps are handled with the averaging scheme. This results in a complete PA flutter solution.

Conclusions

The PA flutter method uses the piecewise quadratic interpolation function commonly used for k-method flutter solutions and concepts from transient flutter and the p–k method. A piecewise flutter equation is defined that utilizes a very good approximation to the aerodynamic forces. The substitution of a scaled Laplace variable for the reduced frequency is the same approximation used for transient flutter with RFA aerodynamics. By defining this piecewise flutter equation, all of the flutter eigenvalues are computed with a fixed number of general matrix eigenvalue solutions. The eigenvalues computed from each of the piecewise flutter equations with a frequency within the range valid for that equation are accepted. This is the same basic concept behind the p–k flutter method.

The iterative nature of the p–k method can be costly in terms of computer resources and can also result in convergence problems. The PA flutter method provides an accurate damping flutter solution without iterating. The piecewise nature of the PA flutter method does create some continuity issues. These are easily handled by averaging the solutions as they transition from one range of reduced frequency to the next.

The PA flutter method allows the computation of accurate damping flutter solutions without iterating. This greatly reduces the computer resources (CPU time) required to obtain the accurate damping flutter solution and enhances its reliability.

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